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PLASMA SURFACE RADIATION

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CONTENTS

Synopsis

A simple model for the radiation from a plasma surface layer is proposed. The intensity of the radiation, its frequency spectrum, and its pressure on the emitting surface are considered.

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1. Introduction

There are three main mechanisms of electromagnetic radiation in a high-temperature plasma. These are: 1) bremsstrahlung due to Coulomb collisions of electrons with massive particles, 2) synchrotron radiation from electrons spiraling in a magnetic field, and 3) Cherenkov radiation. All these radiation processes take place inside a plasma volume.

However, in addition to these, electromagnetic radiation of another type must exist in a plasma. It is the bremsstrahlung from the electrons of a space charge near a plasma surface due to their deceleration in a self-consistent electric field. In the following, we consider the simplest model for such a radiation, its intensity, frequency spectrum, and the pressure which acts on the emitting surface as a result of a radiative reaction.

With regard to the radiative pressure, it should be noticed that usually it is regarded as part of a total pressure, which causes the expansion of the system (e.g., inside stars). However, this is true only for a radiation inside a cavity which, while it reflects from the walls of the cavity, causes an outward pressure. In the case of an open system, the radiation emitted into the outer space acts on the radiating surface with a force of reaction directed inwards. Thus, the momentum carried out with the radiation from the unit area of the surface per unit time is equivalent to a pressure compressing the system.

If a plasma existed radiating like a black body at sufficiently high temperature, the radiative reaction would produce an effect of "walls" preventing the system from expanding. In that case, the radiative pressure would be equal to $\sigma T^4/C$ according to the Stefan-Boltzmann law.

Since the kinetic pressure in a plasma p can be expressed by its temperature kT from the equation of state for the perfect gas $p = nkT$, where n is the numerical density of plasma particles, it is seen that, at $T = 10^5\ ^\circ K$, the radiative reaction would be able to compensate the pressure of a plasma with concentration $n = 10^{16}$ particles/cm^3 (that is enough, for example, if the thermonuclear fusion took place in a deitherium plasma). At a temperature $10^8\ ^\circ K$, the "confinement concentration" is equal to 10^{25} particles/cm^3.

Actually the radiation from a tenuous plasma is considerably less than that of a black body and therefore the contribution of its reaction to the confinement, though positive, is very small. At the same time, in the case of a more dense plasma of stars, the contribution can be rather essential. It is especially important in view of the well-known feature of gravitation that cannot provide the stable existence of a finite hot system *in vacuo*. There is also a chance that, in a future device for controlled thermonuclear fusion, the radiative reaction can help essentially to a main confinement mechanism if the process takes place in a dense plasma and the energy is utilized through the radiation.

2. Situation near a plasma surface

We consider a quasi-neutral plasma in which an electronic charge is compensated on the average by a positive ion background. Like any hot conductor, the plasma must be surrounded with a cloud of a space charge which consists of thermo-emission electrons crossing a plasma surface owing to their thermal motion. These electrons and the plasma surface itself produce an electric field.

To obtain an electrostatic potential of that self-consistent field, we have to solve the Poisson equation with the density of the thermo-electronic space charge in the right-hand side and the boundary condition corresponding to the equality between a surface charge density and the total space charge under unity area (neutrality condition). Since a layer of the space charge is very thin (its thickness has an order of the Debye length), one can restrict oneself to a one-dimensional solution of the Poisson equation that provides an expression for the electric field near the plasma surface independently of a configuration of the system.

With a usual assumption, the electron density of the space charge obeying the Boltzmann distribution law and taking the z-axis along an outward normal to a plasma surface, one has the following solution for the potential $\Psi(z)$[1]:

$$\Psi(z) = -\frac{2\,kT}{e}\,ln\left(\frac{z}{a}+1\right), \tag{1}$$

where kT is the temperature in energy units, e is the electronic charge, $a = \sqrt{kT/2\pi e^2 n_0}$ is the Debye length, n_0 being an electron density at $z = 0$ (at the plasma surface).

This gives the electric field E:

$$E_z = -\partial\Psi/\partial z = 2\,kT/e(z+a),\ E_x = E_y = 0, \tag{2}$$

which acts on the electrons of the space charge and returns them backwards into a plasma.

Such a motion of electrons must be accompanied by an electromagnetic radiation. The power $dW_{\boldsymbol{n}}(t)$ radiated by one charge into the solid angle $d\Omega = \sin\Theta d\Theta\, d\Phi$ in the direction of the unit vector $\boldsymbol{n}$ is[2])

$$\left.\begin{aligned} dW_{\boldsymbol{n}}(t) = \frac{e^2}{4\pi c}\Bigg\{&\frac{2(\boldsymbol{n}\cdot\dot{\beta})(\beta\cdot\dot{\beta})}{(1-\boldsymbol{n}\cdot\beta)^4} + \frac{\dot{\beta}^2}{(1-\boldsymbol{n}\cdot\beta)^3} \\ &- \frac{(1-\beta^2)(\boldsymbol{n}\cdot\dot{\beta})^2}{(1-\boldsymbol{n}\cdot\beta)^5}\Bigg\}\, d\Omega, \end{aligned}\right\} \tag{3}$$

$\beta = \boldsymbol{V}/C$, ∇ and $\dot{\nabla}$ being a velocity and an acceleration of the charge, respectively.

Expressing the acceleration $\dot{\boldsymbol{V}}$ by an electric field $\boldsymbol{E}$ from the relativistic equation of an electron motion and taking into account that the field (2) has a constant direction along the z-axis, we can rewrite (3) in the form

$$dW_{\boldsymbol{n}}(t) = \frac{e^4E^2}{4\pi m^2c^3}\left\{\frac{(1-\beta^2)(1-\beta_z^2)}{(1-\boldsymbol{n}\cdot\beta)^3} - \frac{(1-\beta^2)^2(\cos\Theta-\beta_z)^2}{(1-\boldsymbol{n}\cdot\beta)^5}\right\} d\Omega. \tag{4}$$

Here, E is the Z-component of the field (2) and a polar axis has been taken along the z-axis (i.e., along the direction of the field $\boldsymbol{E}$). With such a choice for the polar axis, we get

$$1-\boldsymbol{n}\cdot\beta = 1-\beta_z\cos\Theta-(\beta_x\cos\Phi+\beta_y\sin\Phi)\sin\Theta. \tag{5}$$

Since we are interested in the total power radiated, we have to take a sum of expressions like (4) for all the charges crossing the unit area of the plasma surface. The electrons cross the surface with different initial velocities, but we consider for a while only the particles which pass through a plasma surface element ΔS having almost equal initial velocities $\boldsymbol{V}_0$ in an interval $d\boldsymbol{V}_0$. These particles are moving in the field (2) inside a narrow current tube, which would have a parabolic shape if the motion were a non-relativistic one and the field $\boldsymbol{E}$ were uniform (projectile motion).

Since electrons in the space charge are supposed to have a stationary distribution, the same number of particles pass any cross section of the tube per unit time. This is just the same number of particles that leaves the plasma and enters the field through ΔS per unit time and returns back in another place having the same magnitude but the opposite sign of the Z-component of the velocity V_z.

To obtain a radiation from the current tube, consider its element of length dl. A volume of the element equals $\Delta S_n dl$, and the number of particles inside is $n\Delta S_n dl$, where n is the electron concentration in the tube and ΔS_n is its normal cross section (both values are varying along the tube). Then, the radiation of the particles is

$$\frac{dW_n'}{d\Omega} = \int \left(\frac{dW_n}{d\Omega}\right)_1 n\Delta S_n dl, \tag{6}$$

where the integrand is the radiation power for one particle (4) and the integral is to be taken along the current tube.

Using the time of the motion t as a parameter and taking into account that $\boldsymbol{dl} = \boldsymbol{V}dt$, we can rewrite the expression (6) as an integral along the path of a particle:

$$\frac{dW_n'}{d\Omega} = \int_0^\tau (dW_n/d\Omega)_1 n\Delta S_n V dt, \tag{7}$$

where τ is the total time of the particle motion in the field $\boldsymbol{E}$.

It is evident that $nv = j$ is a current density in the tube and $j\Delta S_n$ is the total number of particles that cross any section of the tube in unit time. In view of the stationarity,

$$n\Delta S_n V = dN(\boldsymbol{V_0})\Delta S, \tag{8}$$

where $dN(\boldsymbol{V_0})$ is the number of particles hitting ΔS from the plasma interior per unit time with the velocity $\boldsymbol{V}_0$.

The number $dN(\boldsymbol{V_0})$ can be expressed with a velocity distribution function of electrons $f_e(V_0)$, which is regarded as independent of a velocity vector direction. If Θ_0 is an angle of incidence, we have

$$dN(V_0, \Theta_0) = n_0 f_e(V_0) V_0 \cos\Theta_0 2\pi V_0^2 \sin\Theta_0 d\Theta_0 dV_0, \tag{9}$$

where $2\pi V_0^2 \sin\Theta_0 d\Theta_0 dV_0 = V_0^2 dV_0 d\Omega_0$ is a volume element of a velocity space, which has been integrated over an azimuthal angle Φ_0.

After integration (7) over all possible V_0 and all directions Θ_0 corresponding to outward motions of electrons from a plasma, we have the final expression for the total power emitted from the unit area of the plasma surface in the direction of n:

$$\left(\frac{dW_{\boldsymbol{n}}}{d\Omega}\right)_{\text{tot}} = 2\pi n_0 \int_0^\infty V_0^3 f(V_0)\, dV_0 \int_0^{\pi/2} \cos\Theta_0 \sin\Theta_0\, d\Theta_0 \int_0^\tau \left(\frac{dW_{\boldsymbol{n}}}{d\Omega}\right)_1 dt. \tag{10}$$

3. Total power of the surface radiation

Since we are interested in a total energy emitted into the outer space, we have to integrate the expression (10) over all directions of $\boldsymbol{n}$, corresponding to the outer hemisphere. For the power $(dW_{\boldsymbol{n}}/d\Omega)_1$ radiated by one particle, an integration of the expression (4) over the azimuthal angle Φ in $d\Omega$ leads to the integrals

$$\left.\begin{aligned} \int_0^{2\pi} \frac{d\Phi}{(p+q\cos\Phi + r\sin\Phi)^3} &= \pi\frac{2p^2+q^2+r^2}{(p^2-q^2-r^2)^{5/2}}, \\ \int_0^{2\pi} \frac{d\Phi}{(p+q\cos\Phi + r\sin\Phi)^5} &= \pi\frac{8p^4+24p^2(q^2+r^2)+3(q^2+r^2)^2}{4(p^2-q^2-r^2)^{9/2}}, \end{aligned}\right\} \tag{11}$$

where $p = 1-\beta_z\cos\Theta$, $q = \beta_x\sin\Theta$, $r = \beta_y\sin\Theta$.

Inserting the notations $\beta_\tau = \sqrt{\beta_x^2+\beta_y^2}$ and $\cos\Theta = x$ (that is not the Θ_0!), we obtain the following expression for the power radiated from the current tube (7):

$$W' = \frac{e^4}{4m^2c^3}\int_0^\tau E^2(1-\beta^2)\left[\int_0^1 I(x)\,dx\right]dt, \tag{12}$$

where

$$\left.\begin{aligned} I(x) = {} & \frac{2(1-\beta_z^2)}{R^3} + \frac{3\beta_\tau^2(1-x^2)(1-\beta_z^2)}{R^5} \\ & -(1-\beta^2)(x-\beta_z)^2\left[\frac{2}{R^5} + \frac{10\beta_\tau^2(1-x^2)}{R^7} + \frac{35\beta_\tau^4(1-x^2)^2}{4R^9}\right], \end{aligned}\right\} \tag{13}$$

and

$$R = \sqrt{p^2-q^2-r^2} = \sqrt{\beta^2x^2 - 2\beta_z x + 1 - \beta_\tau^2}.$$

If we ignore the influence of the radiative reaction force on the motion of the particles, we can regard their total energy as a constant. This means that their motion during the first half of the total time of motion τ (while they are "raising") coincides with that during the second half (while they are "falling") if we change $V_z \to -V_z$. Thus,

$$\left.\begin{aligned} W' &= \frac{e^4}{4\,m^2c^3}\left[\int_0^{\tau/2} E^2(1-\beta^2)\left\{\int_0^1 I(x,V_z)\,dx\right\}dt \right. \\ &\quad \left. + \int_{\tau/2}^{\tau} E^2(1-\beta^2)\left\{\int_0^1 I(x,V_z)\,dx\right\}dt\right] \\ &= \frac{e^4}{4\,m^2c^3}\int_0^{\tau/2} E^2(1-\beta^2)\left\{\int_0^1 I(x,V_z)\,dx + \int_0^1 I(x,-V_z)\,dx\right\}dt. \end{aligned}\right\} \tag{14}$$

From (13) it is seen that the integrand $I(x,V_z)$ has the following property:

$$I(x,V_z) = I(-x,-V_z), \tag{15}$$

but from this it follows that

$$\int_0^1 I(x,-V_z)\,dx = \int_{-1}^0 I(-x,-V_z)\,dx = \int_{-1}^0 I(x,V_z)\,dx.$$

Taking this into account, we have from (14):

$$W' = \frac{e^4}{4\,m^2c^3}\int_0^{\tau/2} E^2(1-\beta^2)\left\{\int_{-1}^1 I(x,V_z)\,dx\right\}dt. \tag{16}$$

The integral over x in (16) expresses the total intensity of the radiation over all directions and for this we may use the Larmor formula.[2]) In that case, we obtain

$$W' = \frac{2\,e^4}{3\,m^2c^3}\int_0^{\tau/2} \frac{E^2(1-\beta_z^2)}{1-\beta^2}\,dt. \tag{17}$$

Thus, we have the following theorem:

the energy emitted by a charge in a "projectile" motion in a constant electric field into a hemisphere in the direction of the field during the total time of its motion is equal to the total energy emitted during half of this time (ignoring the influence of the radiative reaction on the motion of the charge).

Substituting the expression (2) for the field E into (17) and using the 4-velocity $\boldsymbol{u} = \beta/\sqrt{1-\beta^2}$, we have

$$W' = \frac{2e^2(2kT)^2}{3m^2c^3}\int_0^{\tau/2}\frac{1+u_\tau^2}{(z+a)^2}\,dt. \tag{18}$$

With the aid of the equations of motion for a relativistic electron in the field (2):

$$\frac{du_\tau}{dt} = 0; \quad \frac{du_z}{dt} = -\frac{2kT}{mc(z+a)}, \tag{19}$$

we can come in (18) from the integral over dt to an integral over the z-component of the 4-velocity u_z:

$$W' = \frac{4e^2}{3}\frac{kT}{mc^2}(1+u_\tau^2)\int_0^{u_{0z}}\frac{du_z}{z+a}. \tag{20}$$

Here, we have taken into account that $u_\tau = \text{const}$, in accordance with (19), and also that during the first half of the motion a particle reaches the maximum distance from the plasma surface where the z-component of its velocity u_z becomes zero.

Expressing $(z+a)$ from an energy integral for equations (19),

$$mc^2\sqrt{1+u^2} = mc^2\sqrt{1+u_0^2} - 2kT\,ln\left(\frac{z}{a}+1\right), \tag{21}$$

and inserting it into the integrand of (20), we have

$$W' = \frac{4e^2kT}{3amc^2}(1+u_\tau^2)\int_0^{u_{0z}}\exp\left\{-\frac{mc^2}{2kT}\left(\sqrt{1+u_0^2}-\sqrt{1+u^2}\right)\right\}du_z. \tag{22}$$

After the substitutions

$$u_z = u_{0z}\xi, \qquad u_{0z} = u_0\cos\Theta_0, \qquad u_\tau = u_{0\tau} = u_0\sin\Theta_0,$$

the expression (22) becomes

$$W' = \frac{4\,e^2 kT}{3\,amc^2}(1+u_0^2\sin^2\Theta_0)\,u_0\cos\Theta_0\int_0^1 e^{-\frac{mc^2}{2\,kT}(\sqrt{1+u_0^2}-\sqrt{1+u^2})}\,d\xi, \tag{23}$$

wherein the integrand $u^2 = u_0^2(\sin^2\Theta_0 + \xi^2\cos^2\Theta)$.

To obtain the final expression for the power radiated, we have to integrate (23) in accordance with (10) over all u_0 and Θ_0. In the relativistic case, we ought to take for $f(u_0)$ the Jüttner-Synge distribution function[3])

$$f(u_0) = \frac{mc^2}{4\pi kTK_2(mc^2/kT)}\exp\left\{-\frac{mc^2\sqrt{1+u_0^2}}{kT}\right\}, \tag{24}$$

with the modified Bessel function $K_2\left(\frac{mc^2}{kT}\right)$ of second order.

Then (10) and (23) give

$$\left.\begin{aligned} W &= \frac{2\,e^2 c n_0}{3\,aK_2\left(\frac{mc^2}{kT}\right)}\int_0^\infty u_0^4 du_0\int_0^1 d\xi \\ &\times\int_0^1 [1+u_0^2(1-x_0^2)]\,x_0^2\exp\left\{-\frac{mc^2}{2\,kT}\left(3\sqrt{1+u_0^2}-\sqrt{1+u^2}\right)\right\}dx_0, \end{aligned}\right\} \tag{25}$$

where $x_0 = \cos\Theta_0$.

Considering a weakly relativistic case $u_0 \ll 1$, we have

$$\left.\begin{aligned} &\frac{mc^2}{2\,kT}\left(3\sqrt{1+u_0^2}-\sqrt{1+u^2}\right) \\ &\simeq \frac{mc^2}{kT}\sqrt{1+u_0^2\left(1+\frac{1-\xi^2}{2}x_0^2\right)}+\frac{3\,mc^2(1-\xi^2)^2 x_0^4 u_0^4}{32\,kT}, \end{aligned}\right\} \tag{26}$$

and

$$\exp\left\{-\frac{3\,mc^2 x_0^4(1-\xi^2)^2 u_0^4}{32\,kT}\right\} \simeq 1-\frac{3\,mc^2 x_0^4(1-\xi^2)^2 u_0^4}{32\,kT},$$

so that the expression (25) becomes

$$W = \frac{2\, e^2 c n_0}{3\, a K_2 \left(\frac{mc^2}{kT}\right)} \int\limits_0^1 dx_0 \int\limits_0^1 d\xi \int\limits_0^\infty \Bigg[1 + (1 - x_0^2)\, u_0^2$$

$$- \frac{3\, mc^2 x_0^4 (1 - \xi^2)^2}{32\, kT} u_0^4 \Bigg] u_0^4 x_0^2\, e^{-\frac{mc^2}{kT} \sqrt{1 + u_0^2 \left(1 + \frac{1 - \xi^2}{2} x_0^2\right)}}\, du_0 .$$

Replacement of the variable

$$u_0 \sqrt{1 + \frac{1 - \xi^2}{2} x_0^2} = \eta$$

leads to

$$W = \frac{2\, e^2 c n_0}{3\, a K_2 (mc^2/kT)} \left\{ I_1 + I_2 - \frac{3\, mc^2}{32\, kT} I_3 \right\}, \tag{27}$$

where

$$\left.\begin{aligned} I_1 &= \int\limits_0^1 d\xi \int\limits_0^1 \frac{x_0^2 dx_0}{R^5} \int\limits_0^\infty \eta^4 \exp\left\{ - \frac{mc^2 \sqrt{1 + \eta^2}}{kT} \right\} d\eta , \\ I_2 &= \int\limits_0^1 d\xi \int\limits_0^1 \frac{(1 - x_0^2)\, x_0^2 dx_0}{R^7} \int\limits_0^\infty \eta^6 \exp\left\{ - \frac{mc^2 \sqrt{1 + \eta^2}}{kT} \right\} d\eta , \\ I_3 &= \int\limits_0^1 (1 - \xi^2)^2 d\xi \int\limits_0^1 \frac{x_0^6 dx_0}{R^9} \int\limits_0^\infty \eta^8 \exp\left\{ - \frac{mc^2 \sqrt{1 + \eta^2}}{kT} \right\} d\eta , \\ R &= \sqrt{1 + \frac{1 - \xi^2}{2} x_0^2} . \end{aligned}\right\} \tag{28}$$

The integrals over $x_0 = \cos \Theta_0$ and $\xi = u_z / u_{0z}$ are rather simple and the integrals over η can be expressed by the modified Bessel functions. Thus, we have

$$W = \frac{4\, e^2 c n_0 \alpha^2}{9\, a K_2 (\alpha^{-1})} \left\{ K_3 (\alpha^{-1}) + 2\, \alpha \left[K_4 \left(\frac{1}{\alpha}\right) - \frac{1}{6} K_5 \left(\frac{1}{\alpha}\right) \right] \right\},$$

where $\alpha = kT/mc^2$.

In accordance with our approximation (26), we have to take here the asymptotic expansions for K_ν corresponding $\alpha \ll 1$. Finally, we have

$$W = \frac{4\sqrt{2}\,\pi e^3 (n_0 kT)^{3/2}}{9\,m^2 c^3}\left[1 + \frac{35}{3}\left(\frac{kT}{mc^2}\right)\right] \tag{29}$$

for the total power radiated from the unity area of a plasma surface.

4. Radiative reaction

It has already been pointed out in the introduction that the radiative reaction is equivalent to a pressure compressing the system. The pressure is equal to a normal component of the momentum carried out with the radiation from unit area of the plasma surface per unit time.

In accordance with (7) and taking into account the connection $\boldsymbol{P} = (\varepsilon/c)\,\boldsymbol{n}$ between the energy ε and the momentum $\boldsymbol{P}$ of an electromagnetic wave, we have for the radiation pressure of particles belonging to the current tube (6) by analogy with (12);

$$p' = \frac{e^4}{4\,m^2 c^4}\int_0^{\tau} E^2 (1-\beta^2)\left\{\int_0^1 x I(x)\,dx\right\} dt, \tag{30}$$

where again $x = \cos\Theta$, and for $I(x)$ we have to take the expression (13). Here we also have taken into account that we need only the z-component of a wave's momentum which is equal to $\varepsilon\cos\Theta/c = \varepsilon x/c$.

Like the case of expression (14) we can write again, taking into account a reversibility of the motion:

$$p' = \frac{e^4}{4\,m^2 c^4}\left[\int_0^{\tau/2} E^2 (1-\beta^2)\left\{\int_0^1 x I(x, v_z) + \int_0^1 x I(x, -v_z)\,dx\right\}\right] dt.$$

However, now the integrand $xI(x, V_z)$ does not possess the property (15) and a theorem analogous to that of section 3 does not take place. This makes the calculation rather long, and at last we come to the expression

$$\left.\begin{aligned} p' = \frac{e^4}{3m^2c^4}\int_0^{\tau/2} E^2 &\left\{\frac{2(1-\beta_z^2)\sqrt{1-\beta_\tau^2}}{1-\beta^2} - \frac{1-\beta_z^2}{\sqrt{1-\beta_\tau^2}}\right. \\ &\left. + \frac{(1-\beta_z^2)(1-\beta^2)}{8(1-\beta_\tau^2)\sqrt{1-\beta_\tau^2}} - \frac{3(1-\beta^2)^2}{8(1-\beta_\tau^2)^2\sqrt{1-\beta_\tau^2}}\right\} dt, \end{aligned}\right\} \tag{31}$$

or, expressing this with the 4-velocity $\boldsymbol{u}$,

$$\left.\begin{aligned} p' &= \frac{e^4}{3\,m^2c^4}\int\limits_0^{\tau/2} E^2 \sqrt{\frac{1+u_z^2}{1+u^2}}\left\{2\,(1+u_\tau^2)\right. \\ &\left. -\frac{1+u_\tau^2}{1+u_z^2}+\frac{1}{8}\frac{1+u_\tau^2}{(1+u_z^2)^2}-\frac{3}{8}\frac{1+u^2}{(1+u_z^2)^3}\right\}dt. \end{aligned}\right\} \qquad (32)$$

This expression has to be averaged over all the magnitudes of the initial parameters ($u_0, \cos\Theta_0$) with a corresponding distribution function. We restrict ourselves only to a non-relativistic approximation. Ignoring in (32) all the u_i in comparison with unity, we have

$$p' = \frac{e^4}{4\,m^2c^4}\int\limits_0^{\tau/2} E^2 dt = \frac{3\,W'}{8\,c}, \qquad (33)$$

where W' is the power radiated, for which we have the expression (17) in which we have to ignore β_z^2 and β^2 in comparison with unity.

Equation (33) saves us from repeating the calculation with a distribution function. Using the first term of formula (29), we get the following final expression for the compression of a radiating system:

$$p = \frac{\sqrt{2\pi}}{6}\,\frac{e^3}{m^2c^4}\,(n_0 kT)^{3/2}. \qquad (34)$$

From the expression (34) one can see that the pressure is proportional to the gas pressure nkT in the power $3/2$. At the same time, the factor e^3/m^2c^4 is very small (of an order of 10^{-16} in $C\,G\,S$ units) and therefore the pressure is negligible in the laboratory. Nevertheless, in cosmic situations it can be essential, because with it the Boltzmann law in a gravitational field can give us the finite size for a radiating system (it is known that the Boltzmann law with gravitational force only leads to an infinite size for a hot system).

5. Surface radiation spectrum

To detect the surface radiation experimentally one has to know its frequency spectrum. For one particle the energy radiated per unit solid angle per unit frequency interval is[2])

$$\frac{dI'(\omega)}{d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int_0^{\tau} \frac{\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{n}\cdot\vec{\beta})^2}\, e^{i\omega[t-\boldsymbol{nr}(t)/c]}\,dt\right|^2, \tag{35}$$

where the integral is spread over the trajectory $\boldsymbol{r} = \boldsymbol{r}(t)$ of the motion.

To find the spectrum of the total surface radiation, one has to multiply the expression (35) by the number of particles (9) that pass through the unit area of the plasma surface per unit time and integrate then over all the initial velocities V_0 and over all solid angles $d\Omega$ and $d\Omega_0$, corresponding to outward directions for the radiation and outgoing particles.

However, for the calculation of the integral (35), one needs a solution of the relativistic equations of motion (19) that is not expressible in a closed form. To simplify the problem, we can change the real field (2) in which the motion of electrons takes place with a uniform field E^* independent of the coordinate Z. Then, we have to choose E^* in such a way that the result for a total power radiated would be as close as possible to the correct expression (29).

If we calculate the power radiated, assuming the electrons to move in a uniform field E^*, we get

$$W = \frac{ce^3 E^* n_0 \alpha^2}{3\,kTK_2(\alpha^{-1})}\{K_3(\alpha^{-1}) + 2\,\alpha K_4(\alpha^{-1})\}, \tag{36}$$

where again $\alpha = kT/mc^2$.

In contradistinction to the result (28), this expression is precise, because in yielding it we need not make any approximation like (26) or so on. Comparing (36) with (29), one concludes that, if we put

$$E^* = \frac{4\,kT}{3\,e\alpha} = \frac{2}{3}E_{\max}, \tag{37}$$

where $E_{\max} = E(0)$ in the formula (2), we will have almost the same result for W with E^* as with (2) (with an error within 3 $^0/_0$).

Although the calculation becomes simpler now, we are still not able to perform the Fourier analysis in a relativistic case in accordance with the expression (35). Therefore, we consider only the non-relativistic limit, when (35) becomes

$$\frac{dI'(\omega)}{d\Omega} = \frac{e^2}{4\pi^2 c^3}\left|\int_0^{\tau} \boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{V}})\,e^{i\omega t}\,dt\right.. \tag{38}$$

In the case of a uniform field E^*, we get for the radiation of one particle

$$I'(\omega) = \frac{2\,e^2 E^*}{3\,\pi\, c^3 \omega^2}\,(1 - \cos\omega\tau), \tag{39}$$

where $\tau = 2\,m V_{0z}/eE^*$ is the total time of electron motion in a field E^* and an integration has been performed in (39) over solid angles, corresponding to the radiation leaving the system.

To obtain a spectrum of the total radiation, we have to average (39) with the aid of the Maxwell distribution function

$$I(\omega) = \frac{4\,(eE^*)^2}{3\,\pi\, c^3 \omega^2}\sqrt{\frac{m}{2\,\pi k T}}\int_0^\infty V_{0z}\sin^2\left(\frac{m\omega V_{0z}}{eE^*}\right) e^{-\frac{mV^2_{0z}}{2\,kT}} dV_{0z}\,. \tag{40}$$

The integral in (40) can be expressed with a confluent hypergeometric function $F(\alpha, \gamma, z)$.

After substitution of the expression (37) for the effective field E^* and using the Kummer transformation to transform $F(\alpha, \gamma, -z)$ into $F(\gamma - \alpha, \gamma, z)$, we have the following final expression for the spectrum:

$$I(\omega) = \frac{16\,kT(n_0 e)^2}{27\,c^3\omega^2}\sqrt{\frac{8\,kT}{\pi m}} \times \left\{1 - e^{-\omega^2/\omega^2_0} F\left(-\frac{1}{2}, \frac{1}{2};\ \omega^2/\omega_0^2\right)\right\}, \tag{41}$$

where $\omega_0 = \sqrt{16\,e^2 \pi n_0/9\,m}$ has an order of a reciprocal time that a particle with the thermal velocity $\sqrt{kT/m}$ spends in travelling a distance equal to the Debye length $a = \sqrt{kT/2\,\pi e^2 n_0}$.

The function $I(\omega)$ decreases steadily from its maximum value when $\omega = 0$:

$$I_{\max}(\omega) = I(0) = \frac{32\,kT(n_0 e)^2}{27\,c^3}\sqrt{\frac{8\,kT}{\pi m}}$$

to zero when $\omega \to \infty$, so as for large ω it tends to zero like $(1/\omega^2)$.

6. Discussion

The mechanism of the generation of surface radiation is a rather general one. Such a radiation must take place not only in the case of a plasma but from any hot system, if only the system contains free electrons. The power of the radiation in any case will be of the same form as (29), though maybe with another factor.

A characteristic feature of the surface radiation is its dependence on the temperature T and the electron density n in the combination nkT. In contradistinction to the usual bremsstrahlung due to Coulomb collisions[4]), the surface radiation has a stronger dependence on the temperature (power 3/2 instead of 1/2).

Our expressions (29), (34), and (41) are valid independent of the plasma configuration and the confinement mechanism. This is so because only a very thin layer of a space charge is responsible for the surface radiation.

Since we have ignored the influence of the force of radiative reaction on the particle motion, all our theory is a zeroth approximation in $2e^2/3mC^3$, the "noncausality" time. Nevertheless this does not prevent us from evaluating the radiative reaction pressure (34) in that approximation.

In obtaining the expressions (29), (34), and (41), we have assumed that only the radiation directed outwards leaves the plasma, and the part of the total radiation that falls backwards into the plasma has been absorbed. If the reflection coefficient for this radiation differs from zero, its intensity and pressure reaction would be larger than those given by the formulae (29) and (34). Thus, our expressions give the lower limit for the respective values. At the same time it must be pointed out that the radiation from the surface layer which is directed inwards cannot be considered macroscopically, and therefore neglecting its reflection is not equivalent to the assumption of the plasma being a black body.

Finally, we have to notice that our formulae are valid only if the energy emitted is compensated by its storage, e.g. as a result of some reactions, because our considerations imply an equilibrium. Moreover, the energy transfer in the plasma interior has to be fast enough, otherwise the radiation will cause a cooling of the surface layer.

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